KU Mathematics and Statistics Competition
Department of Mathematics
University of Kansas

Competition for 9th-12th Grades
April 2, 2022

INSTRUCTIONS:

- You have **40 minutes** for the **five problems**.
- Show all of the necessary work and **provide** a complete **justification for each answer**.
- Solve **each problem on a separate sheet** of paper.
- Enclose each **final answer in a box**.
- You are allowed to use a calculator.
- You are not allowed to borrow or interchange calculators during the test.

Problem 1. We call an integer *very odd* if all digits of this number are odd. For example, both 1 and 1339 are very odd, but 1035 is not very odd. Suppose we randomly choose an integer from 1 to 2022 (including 1 and 2022), what is the probability that this integer is very odd?

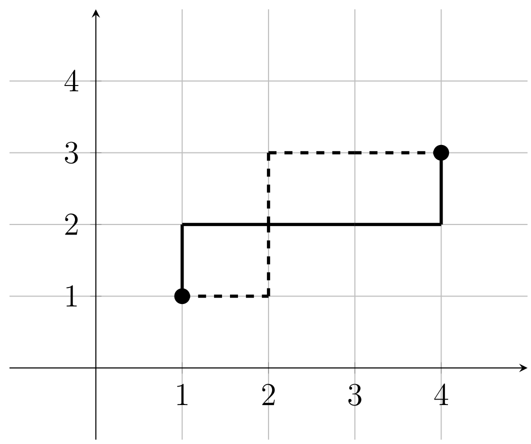
Problem 2. A deck contains 50 cards split into suits of 5 different colors: red, blue, orange, yellow, and green. Each suit has 10 cards, numbered 1 to 10.

Two players can play a card game with this deck, where player A draws two cards and reveals one, then player B draws two cards and says either "higher" if they think the sum of the numbers on their two cards is bigger than the sum of player A's cards, or "lower" if they think the sum is lower. Player B wins if either he is right about his guess or if both hands of players A and B sum to the same number.

If player A reveals a red 6 card, and player B draws a green 4 card and a red 4 card, what is the probability that player B wins this game by guessing "lower"?

Problem 3. The polynomial $1 - x + x^2 - x^3 + \ldots + x^{10} - x^{11}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \ldots + a_{10} y^{10} + a_{11} y^{11}$, where $y = x + 1$ and all a_i's are constants. Find the value of the coefficient a_2.

Problem 4. In a city with streets forming a grid, distance is measured by the number of blocks to get from one point to another. That is, we define a distance function $e(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. If we assume that all x_1, x_2, y_1, y_2 are integers, $e(P_1, P_2)$ measures the smallest number of steps from P_1 to P_2 when traveling up or down, left or right along the xy-grid of length 1. See the example below, which shows the points $P_1 = (1, 1)$ and $P_2 = (4, 3)$, and highlights two paths of length $5 = e(P_1, P_2)$.



The distance function $e(P_1, P_2)$ is sometimes called the **taxicab distance** from point P_1 to point P_2. Using the taxicab distance, find the *taxicab circle* centered at the point $C = (-1, 2)$ of radius $r = 3$, that is, give all integer points $P = (a, b)$ where $e(C, P) = 3$.

Note: Your answer should be a list of the points or a grid with the points plotted on it.

Problem 5. Let S_1 be the set of all possible remainders you may obtain when you divide the sequence of numbers $1,\ 1 \times 10,\ 1 \times 10^2,\ 1 \times 10^3, \ldots$ by 13. Let n be the number of elements in the set S_1. Likewise, for each $i = 1, 2, \ldots, 12$, define S_i to be the set of all possible remainders when you divide the numbers $i,\ i \times 10,\ i \times 10^2,\ i \times 10^3, \ldots$ by 13. Some of the sets $S_1,\ S_2,\ \ldots,\ S_{12}$ are the same (recall that two sets are the same if they contain the same number of elements and all the elements are the same.) Let m be the number of distinct sets among $S_1,\ \ldots,\ S_{12}$. What is the value of the product $n \times m$?