KU Mathematics and Statistics Competition
Department of Mathematics
University of Kansas

Competition for 9th-12th Grades
April 10, 2021

INSTRUCTIONS:

- You have **40 minutes** for the **five problems**.
- Show all of the necessary work and **provide a complete justification for each answer**.
- Solve **each problem on a separate sheet of paper**.
- Enclose each **final answer in a box**.
- You are allowed to use a calculator.

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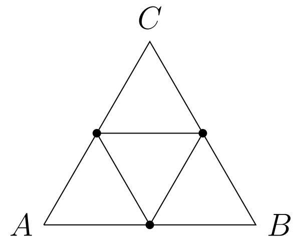
Problem 1. Let $\triangle ABC$ be an equilateral triangle, i.e., a triangle in which all three sides are equal, with side lengths

$$|AB| = |BC| = |CA| = \frac{1}{2}.$$

Inscribe within $\triangle ABC$ another triangle whose vertices are the midpoints of the sides of $\triangle ABC$. This divides $\triangle ABC$ into four equilateral triangles. Repeat this process with the inscribed triangle to further divide it into four more equilateral triangles. Continuing in this manner, what is the total area of the inscribed triangles?



Problem 2. The *Fibonacci* numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$. After the first two Fibonacci numbers, each Fibonacci number is equal to the sum of the two before it, i.e.,

$$F_1 = 1, \qquad F_2 = 1, \qquad F_3 = F_2 + F_1 = 2, \qquad F_4 = F_3 + F_2 = 3,$$

and generally $$F_i = F_{i-1} + F_{i-2}, \qquad \text{for} \quad n \geq 3.$$

If F_{2020} is the 2020^{th} Fibonacci number, find the remainder when $\left(F_{2020}\right)^{F_{2020}}$ is divided by 3.

Problem 3. Lauren tosses a ring of radius 1 at a square of side length 6 on the ground. She wins if the ring lands entirely within the square. If the center of the ring is uniformly distributed within the square, what's the probability that Lauren wins?

Problem 4. Given that a Pythagorean triple consists of three positive integers a, b and c satisfying

$$a^2 + b^2 = c^2,$$

write b and c as a function of a for the following sequence of Pythagorean triples

$$3, 4, 5 \qquad 5, 12, 13 \qquad 7, 24, 25 \qquad 9, 40, 41 \qquad 11, 60, 61 \ldots$$

Problem 5. When the mean, median and mode of the list

$$20, 4, 10, 4, 8, 4, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What are **all** the possible **real values of** x?