Vizing’s Conjecture and Techniques from Computer Algebra

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Definition of Dominating Set Problem

**Dominating Set**: Given a graph $G$ and an integer $k$, does there exist a subset of vertices $D$, with $|D| = k$, such that every vertex in the graph is in, or adjacent to, a vertex in $D$?
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![Graph](image)
Cartesian Product Graph, $G \square H$

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Given vertices $iu, jv \in V(G \square H)$, there is an edge between $iu$ and $jv$ if $i = j$ and $(u, v) \in E[H]$, or $u = v$ and $(i, j) \in E[G]$. 
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```
  G
  2 3

  H
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  b
```
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Vizing's Conjecture (1963)

Given graphs $G$ and $H$,

$$\gamma(G)\gamma(H) \leq \gamma(G \square H).$$
Brief History of Progress

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- In 1990, Faudree, Schelp and Shreve prove that Vizing’s conjecture holds for graphs that satisfy a special “coloring property”.
- In 1991, El-Zahar and Pareek show that Vizing’s conjecture holds for cycles.
- In 2000, Clark and Suen show that $\gamma(G) \gamma(H) \leq 2 \gamma(G_2 H)$.
- In 2003, Sun proves that Vizing’s conjecture holds if $\gamma(G) \leq 3$. 
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An arbitrary graph $G$ in $n$ vertices and a dominating set of size $k$

**Lemma**

The following zero-dimensional system of polynomial equations has a solution if and only if there exists a graph $G$ in $n$ vertices that has a dominating set of size $k$.

\[
x_i^2 - x_i = 0, \quad \text{for } i = 1, \ldots, n,
\]
\[
e_{ij}^2 - e_{ij} = 0, \quad \text{for } i, j = 1, \ldots, n \text{ with } i < j,
\]
\[
(1 - x_i) \prod_{\substack{j=1 \atop j \neq i}}^{n} (1 - e_{ij} x_j) = 0, \quad \text{for } i = 1, \ldots, n,
\]
\[
-k + \sum_{i=1}^{n} x_i = 0.
\]
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Let $S^k_n$ denote the set of $k$-subsets of $\{1, 2, \ldots, n\}$.
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\prod_{S \in S^k_n} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}) \right) \right) = 0.
\]

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e^{2}_{ij} - e_{ij} = 0, \quad \text{for } 1 \leq i < j \leq n,
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Let $\mathcal{P}_G$ be the set of polynomials representing a graph $G$ in $n$ vertices with a dominating set of size $k$:

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Let $\mathcal{P}_H$ be the set of polynomials representing a graph $H$ in $n'$ vertices with a dominating set of size $l$:

\[
e_{ij}'^2 - e_{ij}' = 0 \text{, for } 1 \leq i < j \leq n',
\]

\[
\prod_{S \in S^l_{n'}} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}') \right) \right) = 0.
\]
Let $\mathcal{P}_{G \Box H}$ be the set of polynomials representing the cartesian product graph $G \Box H$ with a dominating set of size $r$:

For $i = 1, \ldots, n$ and $j = 1, \ldots, n'$,

$$z_{ij}^2 - z_{ij} = 0,$$

$$(1 - z_{ij}) \prod_{k=1}^{n} (1 - e_{ik}z_{kj}) \prod_{k=1}^{n'} (1 - e'_{jk}z_{ik}) = 0,$$

and

$$-r + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} = 0.$$
The ideal $I_k^l$ and variety $V_k^l$

**Lemma**

The system of polynomial equations $P_G, P_H$ and $P_{G \Box H}$ has a solution if and only if there exist graphs $G, H$ in $n, n'$ vertices respectively with dominating sets of size $k, l$ respectively such that their cartesian product graph $G \Box H$ has a dominating set of size $r$. 

Let $I_k^l := I(V_k^l)$. 

Note that $I(V_k^l) = I_k^l$ since the ideal $I_k^l$ is radical. 

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Let $I_k^l := I(n, k, n', l, r = kl - 1) := \langle \mathcal{P}_G, \mathcal{P}_H, \mathcal{P}_{G \Box H} \rangle$. 
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**Lemma**

The system of polynomial equations $\mathcal{P}_G, \mathcal{P}_H$ and $\mathcal{P}_{G \square H}$ has a solution if and only if there exist graphs $G, H$ in $n, n'$ vertices respectively with dominating sets of size $k, l$ respectively such that their cartesian product graph $G \square H$ has a dominating set of size $r$.

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Unions and Vizing’s Conjecture

Theorem

Vizing’s conjecture is true $\iff V_{k-1}^l \cup V_{k-1}^{l-1} = V_k^l$. 
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Proof.

Every point in the variety corresponds to a \( G, H \) pair.
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Thus, Vizing’s conjecture is true $\iff V_{k-1}^l \cup V_{k}^{l-1} = V_{k}^l$.  

Intersections and Vizing’s Conjecture

Corollary

Vizing’s conjecture is true $\iff I_{k-1}^l \cap I_{k-1}^{l-1} = I_k^l$. 
**Definition:** Given \( I = \langle f_1, \ldots, f_s \rangle \) and \( J = \langle g_1, \ldots, g_t \rangle \), then the **product ideal** \( I \cdot J := \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle \).
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**Fact:** Given radical ideals $I, J \in \mathbb{K}[x_1, \ldots, x_n]$, $\sqrt{I \cdot J} = I \cap J$. 
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Corollary

Vizing's conjecture is true $\iff I_{l-1}^I \cdot I_{k-1}^I + \langle e_i^2 - e_i, e_j'^2 - e_j', z_{ij}'^2 - z_{ij} \rangle = I_k^I$.
Corollary

Vizing’s conjecture is true ⇐⇒

\[ l_{k-1}^l \cdot l_{k}^{l-1} + \langle e_i^2 - e_i, e_j'^2 - e_j', z_{ij}^2 - z_{ij} \rangle = l_k^l \]

Let

\[ P_{G_{k-1}} := \prod_{S \in S_{n}^{k-1}} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}) \right) \right), \]

\[ P_{H_{l-1}} := \prod_{S \in S_{n'}^{l-1}} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}') \right) \right). \]
Vizing’s Conjecture and Linear Algebra

Corollary

Vizing’s conjecture is true \iff
\[ I_{k-1}^l \cdot I_{k}^{l-1} + \langle e_i^2 - e_i, e'_j^2 - e'_j, z_{ij}^2 - z_{ij} \rangle = I_k^l. \]

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Since \( V_{k-1}^l \cup V_{k}^{l-1} \subseteq V_k^l \), this implies \( I_k^l \subseteq I_{k-1}^l \cap I_{k}^{l-1} \).
Corollary

Vizing’s conjecture is true \iff

\begin{align*}
l^l_{k-1} \cdot l^{l-1}_k + \langle e_i^2 - e_i, e'_j^2 - e'_j, z_{ij}^2 - z_{ij} \rangle &= l^l_k.
\end{align*}

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P_{G_{k-1}} &:= \prod_{S \in S_{n}^{k-1}} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}) \right) \right), \\
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Corollary

Vizing’s conjecture is true $\iff$

$$I_{k-1}^l \cdot I_{k-1}^{l-1} + \langle e_i^2 - e_i, e_j'^2 - e_j', z_{ij}^2 - z_{ij} \rangle = I_k^l.$$ 

Let

$$P_{G_{k-1}} := \prod_{S \in S_n^{k-1}} \left( \sum_{i \notin S} \left( \prod_{j \in S} (1 - e_{ij}) \right) \right),$$

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Since $V_{k-1}^l \cup V_{k-1}^{l-1} \subseteq V_k^l$, this implies $I_k^l \subseteq I_{k-1}^l \cap I_{k-1}^{l-1}$.

Proving $I_k^l \subseteq I_{k-1}^l \cap I_{k-1}^{l-1}$ is equivalent to proving

$$P_{G_{k-1}} \cdot P_{H_{l-1}} \in I_k^l.$$
Let

\[ P'_{G \Box H} := P_{G \Box H} \setminus \left\{ -(kl - l) + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} \right\} \]
Let

\[ \mathcal{P}'_{G \square H} := \mathcal{P}_{G \square H} \setminus \left\{ - (kl - l) + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} \right\} \]

**Conjecture via Experimental Observation**

The following set of polynomials (described by cases 1 through 6) is a graph-theoretic interpretation of the unique, reduced Gröbner basis of \( \mathcal{P}'_{G \square H} \).
Vizing's Conjecture and Gröbner Bases: Degree

Every polynomial in the Gröbner basis has the following form:
\[(x_i^1 - 1)(x_i^d - 1) \cdots (x_i^D - 1),\]
where
\[D = (n - 1) + (n' - 1) + 1 = n + n' - 1.\]

In the \(P_3 \square P_2\) example, the degree equals five.
Every polynomial in the Gröbner basis has the following form:

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where $D := (n - 1) + (n' - 1) + 1 := n + n' - 1$. 
Every polynomial in the Gröbner basis has the following form:

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where \(D := (n - 1) + (n' - 1) + 1 := n + n' - 1.\)

In the \(P_{\text{tri} \square \text{tri}}\) example, the degree equals five.
**Notation:** Let \( G \) represent the set of \( G \)-levels in \( G \square H \). Given a level \( l \in G \), let

\[
p(l) := \prod_{i \in V(l)} (x_i - 1) .
\]
Notation: Let $G$ represent the set of $G$-levels in $G\Box H$. Given a level $l \in G$, let

$$p(l) := \prod_{i \in V(l)} (x_i - 1).$$

Example: Consider the $a$-level in $\text{tri} \Box \text{tri}$. Then,

$$p(a) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1).$$
Case 1: There are $|G| \cdot |H|$ polynomials of the form:

$$p(g) \cdot \prod_{l \in G : l \neq g} (x[l_i] - 1),$$

for each $i \in V(G)$ and each level $g \in \mathcal{G}$. 
Case 1: There are \(|G| \cdot |H|\) polynomials of the form:

\[ p(g) \cdot \prod_{\substack{l \in G: \ l \neq g \wedge \ l \in l_i}} (x[l_i] - 1), \quad \text{for each } i \in V(G) \text{ and each level } g \in G. \]

Example: For \(g = a\)-level and \(i = 1\), then

\[ (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1)(z_{1c} - 1) \]
**Notation:** Let $e \in E[H]$. In $G \Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \Box H$. 
**Notation:** Let $e \in E[H]$. In $G \square H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \square H$. In particular, let $h(e)$ define the $G$-level that where the edge originates (according to the lexicographic order), and let $t(e)$ denote the $G$-level where the edge terminates.
Vizing’s Conjecture and Gröbner Bases: Case 2

**Notation:** Let $e \in E[H]$. In $G \Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \Box H$. In particular, let $h(e)$ define the $G$-level that where the edge originates (according to the lexicographic order), and let $t(e)$ denote the $G$-level where the edge terminates.

**Example:** Consider the edge $e'_{ac}$ and the $c$-level in tri $\Box$ tri. Then,

$$p(h(e)) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1),$$

$$p(t(e)) := (z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1).$$
**Case 2:** There are $2||H|| \cdot |G| + 2||G|| \cdot |H|$ polynomials of the following form:

$$(x_e - 1)p(h(e)) \prod_{g \in G: g \not\in G[t(e)] \text{ and } g \not\in G[h(e)]} (g_i - 1), \quad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

$$(x_e - 1)p(t(e)) \prod_{g \in G: g \not\in G[t(e)] \text{ and } g \not\in G[h(e)]} (g_i - 1), \quad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$
Vizing’s Conjecture and Gröbner Bases: Case 2

Case 2: There are $2||H|| \cdot |G| + 2||G|| \cdot |H|$ polynomials of the following form:

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$$(x_e - 1)p(t(e)) \prod_{g \in \mathcal{G}: g \not\in \mathcal{G}[t(e)] \text{ and } g \not\in \mathcal{G}[h(e)]} (g_i - 1), \quad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

Example: For $e = e'_{ac}$ and $i = 1$, then

$$(e'_{ac} - 1)(z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1),$$

$$(e'_{ac} - 1)(z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1)(z_{1b} - 1).$$
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- Code it up and check it!
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Thank you for your kind attention!
Questions, comments, thoughts and suggestions are most welcome.