An improper integral is a definite integral (A) over infinite intervals (**infinite domain**), or (B) with an integrand which has **infinite discontinuity**.

When integrating with unbounded limits of integration, rewrite the improper integral as the limit of proper integrals.

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx\,.$$

If f has an infinite discontinuity at $x = c$ where c lies in $[a, b]$, then

$$\int_a^b f(x)dx = \left(\lim_{t\to c^-} \int_a^c f(x)dx\right) + \left(\lim_{t\to c^+} \int_t^b f(x)dx\right)\,.$$

An important example: $\int_1^\infty \frac{1}{x^p} dx$ converges when $p > 1$ and diverges when $p \leq 1$.

$\int_0^1 \frac{1}{x^p} dx$ converges when $p < 1$ and diverges when $p \geq 1$.

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. (a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent. (b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Practice Problems

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

   (a) $\displaystyle\int_0^1 e^{-x}\,dx$

   (b) $\displaystyle\int_0^\pi \sec(x)\,dx$

   (c) $\displaystyle\int_0^\infty \sin(x)\,dx$

   (d) $\displaystyle\int_0^1 \frac{1}{\sqrt{3-x^2}}\,dx$

   (e) $\displaystyle\int_0^3 \ln|x|\,dx$

2. Show that $\displaystyle\int_1^\infty \frac{1}{x^3+4x+2}\,dx$ converges by comparing it to $\displaystyle\int_1^\infty x^{-3}\,dx$.

3. Evaluate the improper integrals:

   (a) $\displaystyle\int_{-4}^0 (x+2)^{-\frac{1}{3}}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   (c) $\displaystyle\int_0^\infty \sin(x)\,dx$

   (d) $\displaystyle\int_0^{\frac{\pi}{2}} \tan(x)\,dx$

   (e) $\displaystyle\int_0^1 \ln|x|\,dx$

   (f) $\displaystyle\int_{-\infty}^\infty xe^x\,dx$