Section 6.4

Arclength

- Arclength of Parametric Curves,
- Arclength of Functions,
- Surface Area of Solids of Revolution.
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1. Approximate the quantity by a sum of $N$ terms.
2. Pass to the limit as $N \to \infty$.

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In this section we will represent arclength and surface area as integrals.

**Smooth Curve:** A curve \( c(t) = (x(t), y(t)) \) where \( x'(t) \) and \( y'(t) \) exist, are continuous, and are not simultaneously zero.

**Arclength:** The arclength of a smooth curve is the length of the curve once it has been straightened.
Calculating Arclength of \( c(t) = (x(t), y(t)) \)

(i) Subdivide \([a, b]\) into \(N\)-subintervals of length \(\Delta t = \frac{b-a}{N}\).

(ii) Each \(t_i = a + i\Delta t\) corresponds to a point \(P_i = (x_i, y_i) = (x(t_i), y(t_i))\).
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(iii) The arclength is approximated by summing the distances between the $P_i$’s.

\[
\text{Arclength} \approx \sum_{i=1}^{N} |P_{i-1} P_i|
\]

(iv) Using the Mean Value Theorem, it can be shown that

\[
|P_{i-1} P_i| = \sqrt{x'(t_i)^2 + y'(t_{\ast})^2} \Delta t
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(v) Letting $N \to \infty$ we find the arclength of the curve:

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\int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2} \, dt
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**Solution:** We parameterize the circle by letting $x(t) = R \cos(t)$ and $y(t) = R \sin(t)$ on the interval $[0, 2\pi]$; note that this is one amongst an infinite number of parametrizations.
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**Circumference**

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\int_0^{2\pi} \sqrt{(-R \sin(t))^2 + (R \cos(t))^2} \, dt = \int_0^{2\pi} R \, dt = 2\pi R
\]
Example (2): Calculate the arclength of the curve $c(t) = (t^3 + 1, t^2 - 3)$ on the interval $[0, 1]$. 

Solution: We have a parametrization of the smooth curve, though to calculate the integral we will need trigonometric substitution.

$$
\int_{0}^{1} \sqrt{9t^4 + 4t^2} \, dt = \frac{8}{9} \int_{\arctan(1.5)}^{0} \tan(\theta) \sec^3(\theta) \, d\theta = \frac{8}{9} \int \sqrt{1 + u^2} \, du = \frac{1}{27} \left(13\sqrt{13} - 8\right)
$$

Arclength of a Function

A function $y = f(x)$ can be easily parametrized by the equations $x = t$ and $y = f(t)$. Therefore, a smooth function, where $f'(x)$ exists and is continuous, on the interval $[a, b]$ has arclength

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\int_a^b \sqrt{1 + f'(x)^2} \, dx
\]
iClicker Question 1

The length of the graph of \( y = \sin(x^2) \) from \( x = 0 \) to \( x = 2\pi \) is represented by the integral

(A) \( \int_{0}^{2\pi} (1 + \sin(x^2)) \, dx \)

(B) \( \int_{0}^{2\pi} \sqrt{1 + \sin(x^2)} \, dx \)

(C) \( \int_{0}^{2\pi} \sqrt{1 + (2x \sin(x^2))^2} \, dx \)

(D) \( \int_{0}^{2\pi} \sqrt{1 + (2x \cos(x^2))^2} \, dx \)

(E) \( \int_{0}^{2\pi} \sqrt{1 + (2x \sin(x^2) \cos(x^2))^2} \, dx \)
Find arclength of the curve $c(t) = (\cos(t^2), \sin(t^2))$ on the interval $[0, \sqrt{\pi}]$.

(A) 0  
(B) $\sqrt{\pi}$  
(C) $\pi$  
(D) $\pi^2$  
(E) $2\pi$
Speed of a Particle
The speed of a particle moving along a curve \(c(t)\) is related to arclength. The distance traveled by the particle over the time interval \([a, t]\) is given by the arclength integral

\[
s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2} \, du
\]

Warning: The speed calculated above depends upon the parametrization of the curve. For example, the speed of a particle on \(c(t) = (\cos(t), \sin(t))\) is 1 while the speed of a particle on \(d(t) = (\cos(2t), \sin(2t))\) is 2.
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Speed is the rate of change of distance traveled with respect to time. Using the Fundamental Theorem of Calculus,

$$\text{Speed} = \frac{ds}{dt} = \frac{d}{dt} \left( \int_a^t \sqrt{x'(u)^2 + y'(u)^2} \, du \right) = \sqrt{x'(t)^2 + y'(t)^2}$$
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For example, the speed of a particle on $c(t) = (\cos(t), \sin(t))$ is 1 while the speed of a particle on $d(t) = (\cos(2t), \sin(2t))$ is 2.
Find the time $t$ where the speed of a particle with trajectory $c(t) = (t^3 - t, t^2 + 1)$ is minimized, where $t \geq 0$.

(A) $t = 0$

(B) $t = \frac{1}{3}$

(C) $t = \sqrt{3}$

(D) $t = 9$

(E) $t = \sqrt{\frac{3}{2}}$
Surface Area of Solids of Revolution

Suppose a continuous function \( y = f(x) \) on the interval \([a, b]\) is rotated about the \(x\)-axis. What is the surface area of the resulting solid?
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We will approximate the surface area by:

(i) Subdividing the domain into $N$-subintervals and finding points $P_i = (x_i, f(x_i))$. 

![Diagram showing rotation and surface area approximation](image-url)
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We will approximate the surface area by:

(i) Subdividing the domain into $N$-subintervals and finding points $P_i = (x_i, f(x_i))$.

(ii) The surface area resulting from each subinterval’s revolution can be approximated by taking a line between $P_{i-1}$ and $P_i$ and rotating it about the $x$-axis to form the frustum of a cone.
We calculate the surface area of a cone with base radius $r$ and slant height $l$.

Cutting and unraveling the cone we obtain a sector of a circle, which has surface area

$$\frac{1}{2} l^2 \theta = \frac{1}{2} l^2 \left( \frac{2\pi r}{l} \right) = \pi rl$$
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The surface area of the frustum of the cone:

\[
\pi r_2 (l_1 + l) - \pi r_1 l_1
\]

Using similar triangles and letting \( r = \frac{1}{2} (r_1 + r_2) \) we obtain a simpler formula for the surface area,

\[
\text{Surface Area} = 2\pi rl
\]
The surface area of the solid of revolution which results from rotating the function \( y = f(x) \) on the interval \([a, b]\) about a horizontal axis \( y = c \) is

\[
\int_a^b 2\pi \left( \text{radius} \right) d(\text{segment length}) = \int_a^b 2\pi |c - f(x)| \sqrt{1 + f'(x)^2} \, dx
\]

**Example (3):** Calculate the surface area of a sphere of radius \( R \).

**Solution:**
The function \( f(x) = \sqrt{R^2 - x^2} \) rotated about the \( x \)-axis results in the sphere.

\[
\int_{-R}^R 2\pi \sqrt{R^2 - x^2} \sqrt{1 + \left(-x \sqrt{R^2 - x^2}\right)^2} \, dx
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