Section 6.4
Arclength

- Arclength of Parametric Curves,
- Arclength of Functions,
- Surface Area of Solids of Revolution.

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(1) Approximate the quantity by a sum of N terms.
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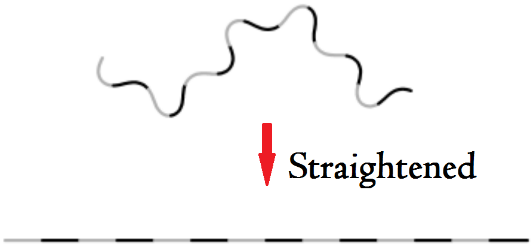
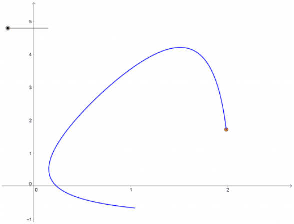
Smooth Curve: A curve $c(t) = (x(t), y(t))$ where $x'(t)$ and $y'(t)$ exist, are continuous, and are not simultaneously zero.



Arclength: The arclength of a smooth curve is the length of the curve once it has been straightened.

Calculating Arclength of $c(t) = (x(t), y(t))$

(i) Subdivide $[a, b]$ into N-subintervals of length $\Delta t = \frac{b-a}{N}$.
(ii) Each $t_i = a + i\Delta t$ corresponds to a point $P_i = (x_i, y_i) = (x(t_i), y(t_i))$.



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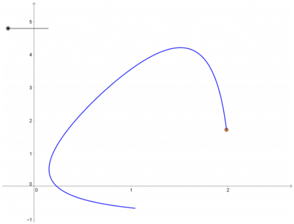

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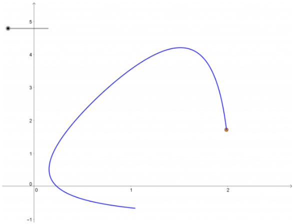

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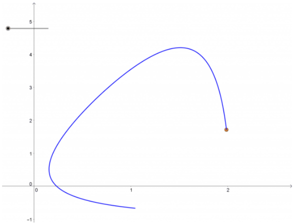

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(v) Letting $N \to \infty$ we find the arclength of the curve:

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Arclength

Let $c(t) = (x(t), y(t))$ be a smooth curve. Then the arclength s of $c(t)$ on the interval $a \leq t \leq b$ is

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Arclength of a Function

A function $y = f(x)$ can be easily parametrized by the equations $x = t$ and $y = f(t)$. Therefore, a smooth function, where $f'(x)$ exists and is continuous, on the interval $[a, b]$ has arclength

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

iClicker Question 1

The length of the graph of $y = \sin(x^2)$ from $x = 0$ to $x = 2\pi$ is represented by the integral

(A) $\displaystyle\int_0^{2\pi} \left(1 + \sin(x^2)\right)\, dx$

(B) $\displaystyle\int_0^{2\pi} \sqrt{1 + \sin(x^2)}\, dx$

(C) $\displaystyle\int_0^{2\pi} \sqrt{1 + (2x\sin(x^2))^2}\, dx$

(D) $\displaystyle\int_0^{2\pi} \sqrt{1 + (2x\cos(x^2))^2}\, dx$

(E) $\displaystyle\int_0^{2\pi} \sqrt{1 + (2x\sin(x^2)\cos(x^2))^2}\, dx$

iClicker Question 2

Find arclength of the curve $c(t) = (\cos(t^2), \sin(t^2))$ on the interval $[0, \sqrt{\pi}]$.

(A) 0

(B) $\sqrt{\pi}$

(C) π

(D) π^2

(E) 2π

Speed of a Particle

The speed of a particle moving along a curve $c(t)$ is related to arclength. The distance traveled by the particle over the time interval $[a, t]$ is given by the arclength integral

$$s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2}\, du$$

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Speed is the rate of change of distance traveled with respect to time. Using the Fundamental Theorem of Calculus,

$$\textbf{Speed} \ = \frac{ds}{dt} = \frac{d}{dt}\left(\int_a^t \sqrt{x'(u)^2 + y'(u)^2}\, du\right) = \sqrt{x'(t)^2 + y'(t)^2}$$

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Warning: The speed calculated above depends upon the parametrization of the curve.

For example, the speed of a particle on $c(t) = (\cos(t), \sin(t))$ is 1 while the speed of a particle on $d(t) = (\cos(2t), \sin(2t))$ is 2.

iClicker Question 3

Find the time t where the speed of a particle with trajectory $c(t) = (t^3 - t, t^2 + 1)$ is minimized, where $t \geq 0$.

(A) $t = 0$

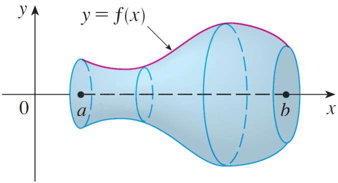
(B) $t = \frac{1}{3}$

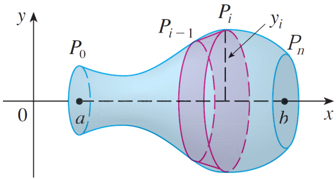
(C) $t = \sqrt{3}$

(D) $t = 9$

(E) $t = \sqrt{\frac{3}{2}}$

Suppose a continuous function $y = f(x)$ on the interval $[a, b]$ is rotated about the x-axis. What is the surface area of the resulting solid?





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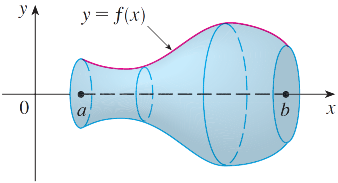

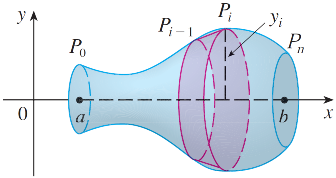

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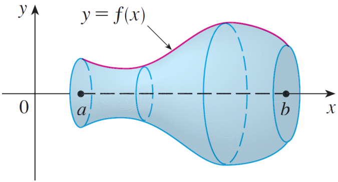

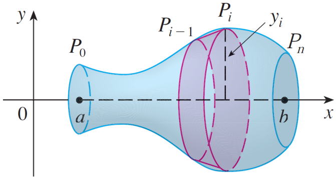

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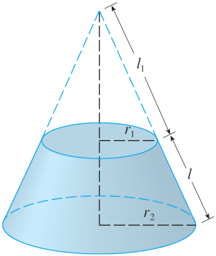
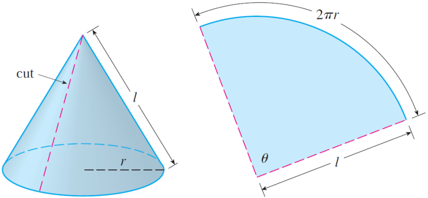
(i) Subdividing the domain into N-subintervals and finding points $P_i = (x_i, f(x_i))$.

(ii) The surface area resulting from each subinterval's revolution can be approximated by taking a line between P_{i-1} and P_i and rotating it about the x-axis to form the frustum of a cone.



We calculate the surface area of a cone with base radius r and slant height l.



Cutting and unraveling the cone we obtain a sector of a circle, which has surface area

$$\frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi r l$$

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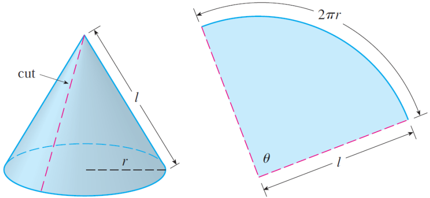

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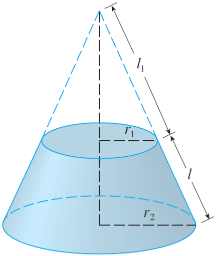
The surface area of the frustum of the cone:

$$\pi r_2(l_1 + l) - \pi r_1 l_1$$

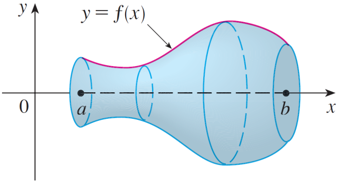
Using similar triangles and letting $r = \frac{1}{2}(r_1 + r_2)$ we obtain a simpler formula for the surface area,

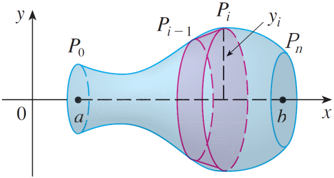
$$\text{Surface Area } = 2\pi r l$$



The surface area of the solid of revolution which results from rotating the function $y = f(x)$ on the interval $[a, b]$ about a horizontal axis $y = c$ is

$$\int_a^b 2\pi\,(\text{ radius })\,d(\text{ segment length }) = \int_a^b 2\pi\,|c - f(x)|\,\sqrt{1 + f'(x)^2}\,dx$$





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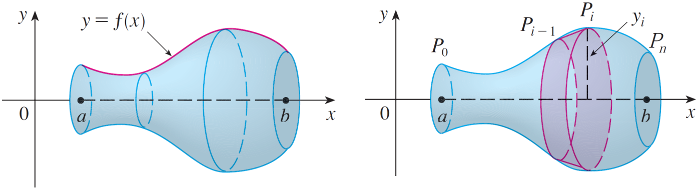

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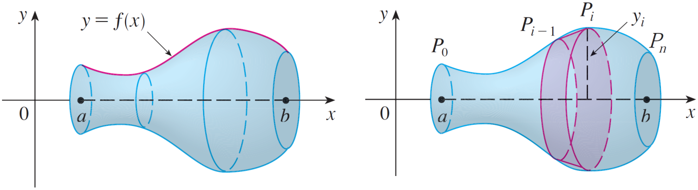

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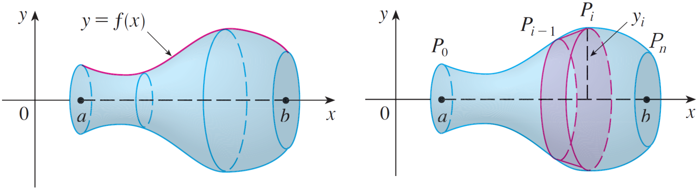

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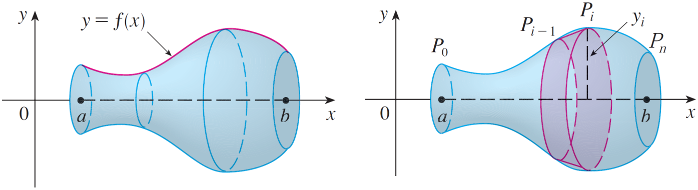

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