Section 6.3
Volume: The Cylindrical Shells Method
A line segment **perpendicular** to the axis of rotation creates a **washer** when rotated. A washer is formed by removing a circle from the center of a larger circle.

**Area:**
\[
\pi r_{\text{outer}}^2 - \pi r_{\text{inner}}^2 = \pi \left( r_{\text{outer}}^2 - r_{\text{inner}}^2 \right)
\]

The distance from the endpoint **nearest** to the axis of rotation is the **inner radius**. The distance from the endpoint **farthest** from the axis of rotation is the **outer radius**.
A line segment parallel to the axis of rotation forms a cylindrical shell when rotated.

The volume of a cylindrical shell is

\[ V = 2\pi \text{(height)} \times \text{(radius)} \times \text{(thickness)} \]
An alternative to the washer method, **cylindrical shells** can be used to calculate the volume of a solid of revolution.

**Shell Method**

(1) Approximate the volume of the solid using $N$ cylindrical shells.
   - Divide $[a, b]$ into $N$-subintervals of length $\Delta x = \frac{b-a}{N}$.
   - Each subinterval generates a shell when rotated.
   - The volume of the solid is approximated by the sum of the shell volumes

\[
\int_{a}^{b} 2\pi \text{(height)} \text{ (radius)} \text{ (thickness)} \, d\text{(thickness)}
\]

(2) The volume equals the limit of the approximations as $N$ approaches infinity.
Example (1): Find the volume of the solid which results from rotating the region enclosed by the curves $y = 1 - 2x + 3x^2 - 2x^3$, $x = 0$, and $y = 0$ about the $y$-axis.

The washer method takes perpendicular cross-sections to the axis which results in a volume integral with respect to $y$. It is difficult to find the inverse of $y = 1 - 2x + 3x^2 - 2x^3$, so it will be difficult to use the washer method. Therefore, we use the Shell Method by taking parallel cuts which results in a volume integral with respect to $x$. A shell has radius $x$ and height $1 - 2x + 3x^2 - 2x^3$.

Volume: $\int_{0}^{1} 2\pi x (1 - 2x + 3x^2 - 2x^3) \, dx$
Example (2): Find the volume of the solid which results from rotating the region enclosed by the curves $y = 1 - 2x + 3x^2 - 2x^3$, $x = 0$, and $y = 0$ about the $x$-axis.

Shell Method

$y = 1 - 2x + 3x^2 - 2x^3$

Washer Method

$y = 1 - 2x + 3x^2 - 2x^3$
Example (3): Find the volume of the solid which results from rotating the region enclosed by the curves \( y = 1 - 2x + 3x^2 - 2x^3 \), \( x = 0 \), and \( y = 0 \) about \( x = 2 \).

This region works best if the integral is with respect to \( x \). When rotating about \( x = 2 \), the Washer Method will use \( y \) while the Shell Method will use \( x \). Therefore, we use the Shell Method by taking parallel cuts which results in a volume integral with respect to \( x \).

The height of a shell is still \( 1 - 2x + 3x^2 - 2x^3 \). The radius of a shell is the distance from the cut to the axis of rotation. Here the radius is \( 2 - x \).
Washer Method

Cross-sections **perpendicular** to the rotational axis are **washers**.

Shell Method

Segments **parallel** to the rotational axis form **cylindrical shells**.