Section 9.4

The Cross Product

(1) The Cross Product,

(2) Area of a Parallelogram,

(3) Triple Product.

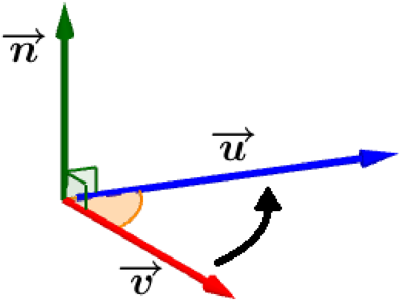
The **cross product** of two vectors $\vec{v}$ and $\vec{u}$, denoted $\vec{v} \times \vec{u}$, is the vector

$$\vec{v} \times \vec{u} = (|\vec{v}|\,|\vec{u}| \sin(\theta))\, \vec{n}$$

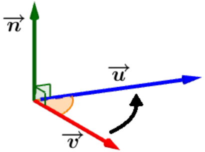
where

(i) θ is the angle between $\vec{v}$ and $\vec{u}$,

(ii) $\vec{n}$ is the unit vector, so that $\vec{n} \perp \vec{v}$, $\vec{n} \perp \vec{u}$, and the vectors $(\vec{v}, \vec{u}, \vec{n})$ satisfies the right-hand rule.

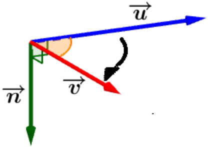


$$\vec{v} \times \vec{u} = (|\vec{v}|\,|\vec{u}|\sin(\theta))\,\vec{n}$$

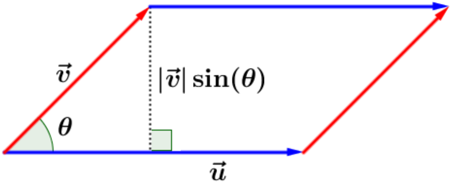


Properties of the Cross Product

- If $\vec{v}$ and $\vec{u}$ are parallel, then $\vec{v} \times \vec{u} = \vec{0}$.
- $(\vec{v} \times \vec{u}) \perp \vec{v}$ and $(\vec{v} \times \vec{u}) \perp \vec{u}$.
- $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$



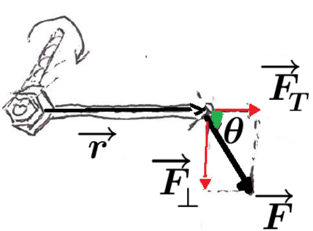
- $|\vec{v} \times \vec{u}|$ is the area of the parallelogram determined by $\vec{v}$ and $\vec{u}$ which is twice the area of the triangle with edges $\vec{v}$ and $\vec{u}$.



If you tighten a bolt by applying a force $\overrightarrow{F}$ to a wrench, it causes the bolt to rotate.

(I) What is the axis about which the bolt rotates?

(II) What is the strength of the rotation?



The answer involves direction and magnitude. It is given by a vector called torque, $\overrightarrow{\tau}$.

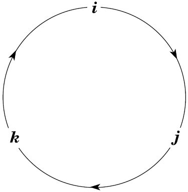
(I) $\overrightarrow{\tau}$ should be orthogonal to $\vec{r}$ and $\overrightarrow{F}$.

(II) $|\overrightarrow{\tau}|$ should be proportional to $|\vec{r}|$ and $\left|\overrightarrow{F}_\perp\right|$.

This motivates the definition of the torque as $\overrightarrow{\tau} = \vec{r} \times \overrightarrow{F}$.

Example: Find the strength of the torque if $\left|\overrightarrow{F}\right| = 20$ (Newtons), $|\vec{r}| = \frac{1}{2}$ meter, and $\theta = \frac{5\pi}{12}$ radians:

$$|\vec{\tau}| = \left|\vec{r} \times \overrightarrow{F}\right| = |\vec{r}|\left|\overrightarrow{F}\right||\sin(\theta)| = 10\sin\left(\frac{5\pi}{12}\right) \approx 9.66 \text{ Joules}$$



$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

Cross Product in Component Form: If $\vec{v} = \langle a_1, b_1, c_1 \rangle$ and $\vec{u} = \langle a_2, b_2, c_2 \rangle$ then we can distribute over the cross product:

$$\vec{v} \times \vec{u} = \left(a_1\vec{i} + b_1\vec{j} + c_1\vec{k}\right) \times \left(a_2\vec{i} + b_2\vec{j} + c_2\vec{k}\right)$$

$$= \begin{array}{c} a_1 b_2 \left(\vec{i} \times \vec{j}\right) + a_1 c_2 \left(\vec{i} \times \vec{k}\right) + a_2 b_1 \left(\vec{j} \times \vec{i}\right) + \\ a_2 c_1 \left(\vec{k} \times \vec{i}\right) + b_1 c_2 \left(\vec{j} \times \vec{k}\right) + b_2 c_1 \left(\vec{k} \times \vec{j}\right) \end{array}$$

$$= (b_1 c_2 - b_2 c_1)\vec{i} - (a_1 c_2 - a_2 c_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

Alternatively, you can consider the determinant of the matrix to calculate the cross product.

The determinant of a 2×2 matrix is calculated:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ac - bd$$

The determinant of a 3×3 matrix can be calculated by decomposing into a linear combination of 2×2 matrices.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

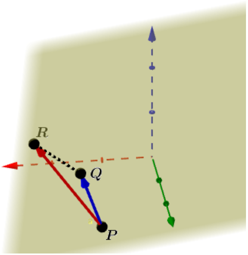
$$\vec{v} \times \vec{u} = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example:

(I) Find a vector which is orthogonal to the plane through the points $P(2, 0, -3)$, $Q(3, 1, 0)$, and $R(5, 2, 2)$.

(II) Find the area of the triangle with vertices at P, Q, and R.

Solution: Let $\vec{u} = \overrightarrow{PR} = \langle 3, 2, 5 \rangle$ and $\vec{v} = \overrightarrow{PQ} = \langle 1, 1, 3 \rangle$.



(I) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix} = \langle 1, -4, 1 \rangle$ is orthogonal to the plane.

(II) The area of the triangle is $\frac{1}{2} |\vec{u} \times \vec{v}| = \frac{3\sqrt{2}}{2}$

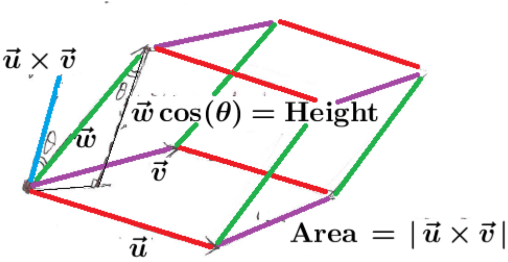
Triple Product

The **triple product** of three vectors $\vec{u}, \vec{v}$, and $\vec{w}$ is defined to be $(\vec{u} \times \vec{v}) \cdot \vec{w}$. It is a scalar given by

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = |\vec{u} \times \vec{v}|\, |\vec{w}| \cos(\theta)$$

where θ is the angle between $\vec{u} \times \vec{v}$ and $\vec{w}$.

Its absolute value $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$ equals the volume of the parallelepiped solid with edges $\vec{u}, \vec{v}$, and $\vec{w}$.



Note: $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}) \cdot \vec{w} = (\vec{w} \times \vec{u}) \cdot \vec{v}$

Triple Product in Components

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{l} a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ -a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{array}$$

Example: Find the volume of the parallelepiped solid with adjacent edges PQ, PR, PS where

$$P(-1, 0, 1) \qquad Q(2, 4, 5) \qquad R(0, 1, 2) \qquad S(6, -1, 4)$$

Solution: We have three vectors which represent the adjacent sides:

$$\vec{u} = \overrightarrow{PQ} = \langle 3, 4, 4 \rangle \quad \vec{v} = \overrightarrow{PR} = \langle 1, 1, 1 \rangle \quad \vec{w} = \overrightarrow{PS} = \langle 7, -1, 3 \rangle$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 3 & 4 & 4 \\ 1 & 1 & 1 \\ 7 & -1 & 3 \end{vmatrix} = -4$$

So the volume of the parallelepiped solid is 4 units^3.