Example 1

By setting \(-x^2 + 8 = 2x + 5\), we get the intersection points

\[ x = -3 \text{ or } x = 1 \]

One stick: \((y_{\text{top}} - y_{\text{bottom}}) \, dx\)

\[ = (-x^2 + 8 - (2x + 5)) \, dx \]

Thus, the area is

\[ \int_{-3}^{1} -x^2 + 8 - (2x + 5) \, dx \]

\[ = -\frac{1}{3}x^3 - x^2 + 3x \bigg|_{-3}^{1} = \frac{32}{3} \]

Example 2

On \([-5, -3]\), \(y_{\text{top}} = 2x + 5\)

\(y_{\text{bottom}} = -x^2 + 8\)

On \([-3, 1]\), the area is given by Example 1

On \([1, 2]\), \(y_{\text{top}} = 2x + 5\), \(y_{\text{bottom}} = -x^2 + 8\)

Thus, the area is

\[ \int_{-5}^{-3} (2x + 5 - (-x^2 + 8)) \, dx + \int_{-3}^{1} (-x^2 + 8 - (2x + 5)) \, dx \]

\[ + \int_{1}^{2} (2x + 5 - (-x^2 + 8)) \, dx \]

\[ = \frac{1}{3}x^3 + x^2 - 3x \bigg|_{-5}^{-3} + \left( -\frac{1}{3}x^3 - x^2 + 3x \right) \bigg|_{-3}^{1} + \left( \frac{1}{3}x^3 + x^2 - 3x \right) \bigg|_{1}^{2} \]

\[ = \frac{71}{3} \]
Example 3:

By setting \( \sin x = \frac{2x}{\pi} \) where \( x \geq 0 \), we get

\[
\int_{0}^{\frac{\pi}{2}} (\sin x - \frac{2x}{\pi}) \, dx = -\cos x - \frac{x^2}{\pi} \bigg|_{0}^{\frac{\pi}{2}} = 1 - \frac{\pi}{4}
\]

Example 4

By setting \( 3y^2 = 8 + y^2 \), we get the intersection points:

\[
y^2 = 4 \quad \text{so} \quad y = \pm 2
\]

Correspondingly,

When \( y = 2 \), \( x = 12 \)

\( y = -2 \), \( x = 12 \)

One "stick" area:

\[
(X_{\text{right}} - X_{\text{left}}) \, dy
= 8 + y^2 - 3y^2 \, dy
\]

Thus

\[
\text{Area} = \int_{-2}^{2} 8 + y^2 - 3y^2 \, dy
= 8y - \frac{2}{3}y^3 \bigg|_{-2}^{2} = \frac{64}{3}
\]
Example 5:

By setting \( \begin{cases} x = y^2 \\ y = x - 2 \end{cases} \), we get the intersection points.

\[ y^2 = y + 2 \]

So \( y = -1 \) or \( y = 2 \).

When \( y = -1 \), \( x = 1 \)

When \( y = 2 \), \( x = 4 \)

Method 1: We use the horizontal sticks. Note that the right "wall" is the line, the left "wall" is the parabola.

\[
\int_{-1}^{2} (X_{\text{right}} - X_{\text{left}}) \, dy = \int_{-1}^{2} (y + 2 - y^2) \, dy = \left. \frac{1}{2} y^2 + 2y - \frac{1}{3} y^3 \right|_{-1}^{2} = \frac{9}{2}
\]

Method 2: We use the vertical sticks. Note that

"Top": the top half of parabola

"Bottom": the lower half of parabola

"Top": the top half of parabola

"Bottom": the line

The "walls" change! We need to find the boundary line.
Thus, we divide the area into two parts:

\( x \) runs from 0 to 1.

\( x \) runs from 1 to 4.

Note that the "walls" don't change in each of the two parts. (pink shaded vs brown shaded)

**Part I:** from \( x = y^2 \), we get \( y = \pm \sqrt{x} \)

\[ y_{\text{Top}} = \sqrt{x} , \quad y_{\text{Bottom}} = -\sqrt{x} \]

Area of Part I = \[ \int_0^1 (y_{\text{Top}} - y_{\text{Bottom}}) \, dx \]

\[ = \int_0^1 2\sqrt{x} \, dx = \frac{4}{3} \cdot \frac{3}{2} \left| _0^1 \right| = \frac{4}{3} \]

**Part II:** \( y_{\text{Top}} = \sqrt{x} , \quad y_{\text{Bottom}} = x - 2 \)

Area of Part II = \[ \int_1^4 (y_{\text{Top}} - y_{\text{Bottom}}) \, dx \]

\[ = \int_1^4 (\sqrt{x} - x + 2) \, dx \]

\[ = \left. \frac{3}{2} x^{\frac{3}{2}} - \frac{1}{2} x^2 + 2x \right| _1^4 = \frac{19}{6} \]

Thus, Area = \( \frac{4}{3} + \frac{19}{6} = \frac{8+19}{6} = \frac{27}{6} = \frac{9}{2} \)

They yield the same answers. However, we prefer the first method in this problem.