1. (a) No. $a_n = (-1)^n$

(b) Yes

(c) No. $a_n = (-1)^n$

2. (a) $\lim\limits_{n\to\infty} 10 + \left(-\frac{1}{9}\right)^n = 10 + \underbrace{\lim\limits_{n\to\infty} \left(-\frac{1}{9}\right)^n}_{\text{Geometric sequence}} = 10 + 0 = 10$

(b) $\lim\limits_{n\to\infty} \arctan\left(\frac{n^2+1}{n}\right) = \arctan(\infty) = \frac{\pi}{2}$

(c) $\lim\limits_{n\to\infty} \ln\left|\frac{2n+1}{3n+4}\right| = \ln\left|\lim\limits_{n\to\infty} \frac{2n+1}{3n+4}\right| = \ln\frac{2}{3}$

(d) $\lim\limits_{n\to\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so $\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$

(e) $\frac{3-4^n}{2+7\cdot 3^n} \leq \frac{3}{2+7\cdot 3^n} - \frac{4^n}{2+7\cdot 3^n}$

$\lim\limits_{n\to\infty} \frac{3}{2+7\cdot 3^n} = \frac{3}{\infty} = 0$

but $\lim\limits_{n\to\infty} \frac{4^n}{2+7\cdot 3^n} = \lim\limits_{n\to\infty} \frac{\frac{4^n}{3^n}}{\frac{2}{3^n}+7} = \frac{\infty}{7} = \infty$

so $\{e_n\}$ Div.

(f) $f_n = \frac{8^{2n}}{n!} = \frac{64^n}{n!} = \frac{64}{n}\cdot\frac{64}{n-1}\cdot \ldots \cdot \frac{64}{1}$

for $n>64$ $= \frac{64}{n}\cdot\frac{64}{n-1}\cdot \ldots \cdot \underbrace{\left(\frac{64}{64}\cdot\frac{64}{63}\cdot \ldots \cdot\frac{64}{1}\right)}_{\text{a fixed number}}$

$\leq \frac{64}{n}\cdot$ (a fixed number)

$\to 0$ as $n\to\infty$

By Squeezing Theorem, $\lim\limits_{n\to\infty} f_n = 0$.

(g) Top : logarithm . Bottom : power. Guess $\lim_{n\to\infty} g_n = 0$

Rigorously, $\lim_{x\to\infty} \frac{(\ln x)^2}{x} \underset{\text{L'Hopital Rule}}{=} \lim_{x\to\infty} \frac{2(\ln x)\cdot\frac{1}{x}}{(x)'} = \lim_{x\to\infty} \frac{2\ln x}{x}$

$= \lim_{x\to\infty} \frac{(2\ln x)'}{x'} = \lim_{x\to\infty} \frac{2}{x} = 0$ so $\lim_{n\to\infty} g_n = 0$

(h) Apply our result in the summary

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} 5^{2n}\left(1-\frac{3}{5n}\right)^{2n} = \infty \cdot e^{-\frac{3}{5}\times 2} = \infty$$

Or

$\lim_{x\to\infty} \left(5-\frac{3}{x}\right)^{2x}$ Consider $\lim_{x\to\infty} \ln\left(5-\frac{3}{x}\right)^{2x}$

$$= \lim_{x\to\infty} \frac{\ln\left(5-\frac{3}{x}\right)}{\frac{1}{2x}} = \frac{\ln 5}{0} = \infty.$$

$$\neq \lim_{x\to\infty} \frac{\left(-\frac{3}{x}\right)'}{\left(5-\frac{3}{x}\right)\left(\frac{1}{2x}\right)'}$$ Be careful.

$\therefore \{h_n\}$ Div.

(i). $\lim_{n\to\infty} n\pi\underbrace{(\cos n\pi)}_{(-1)^n} = \pi\lim_{n\to\infty} n(-1)^n = DNE$ so $\{i_n\}$ Div.

(j). $0 \le \frac{3^n}{2^n+4^n} \le \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \to 0$ as $n\to\infty$.

$\therefore \lim_{n\to\infty} j_n = 0$

3. $a_n = n$

4. $a_n = (-1)^n$, $b_n = (-1)^{n+1}$, so $a_n + b_n = 0$

5. $f(x) = \frac{1}{2x+1}$, $f'(x) = -\frac{2}{(2x+1)^2} < 0$ so $a_n \downarrow$

6. $f(x) = \frac{3x^2}{x^2+1}$, $f'(x) = \frac{6x(x^2+1) - 6x^3}{(x^2+1)^2} = \frac{6x}{(x^2+1)^2} > 0$ for $x \geq 1$

$\therefore$ $b_n \uparrow$

7. $a_n = \frac{1-4^n}{2^n} = \frac{1}{2^n} - \frac{4^n}{2^n} = \frac{1}{2^n} - 2^n$

$n \uparrow$: $\downarrow$ $\uparrow$ so $a_n \downarrow$

$n \to \infty$, $\frac{1}{2^n} = (\frac{1}{2})^n \to 0$, $2^n \to \infty$, so $a_n \to -\infty$.

a_1 is the biggest. $a_1 = \frac{1-4}{2} = -\frac{3}{2}$

so $\{a_n\}_{n=1}^{\infty}$ bounded above by $-\frac{3}{2}$. decreasing, but not bounded below ($a_n \to -\infty$).

8. $a_1 = 1$, $a_2 = -1$, $a_3 = -5$, ...

Guess $a_n < 0$ for $n \geq 2$. By Induction:

(i) True for $n = 2$

(ii) Suppose $a_k < 0$, then $a_{k+1} = 2a_k - 3 < 0$

$\therefore$ $\{a_n\}$ bounded above by $a_1 = 1$

Next, $a_n - a_{n-1} = a_{n-1} - 3 < 0$, so $a_n < a_{n-1}$

$\therefore$ $\{a_n\}$ decreasing.

"Bounded below"? If it were true, we would have $\lim_{n\to\infty} a_n = L$,

then $L = 2L - 3$, so $L = 3$, but $a_n < 0$ for all $n \geq 2$.

Impossible, so $\{a_n\}$ is NOT bounded below.

9. (a). $a_0 = 0 < 2$

Suppose $a_k < 2$, $a_{k+1} = \sqrt{2+a_k} < \sqrt{2+2} = 2$

$\therefore$ $a_n < 2$

(b) $a_0 = 0$, $a_1 = \sqrt{2}$, $a_0 < a_1$

Suppose $a_{k+1} > a_k$.

$a_{k+2} = \sqrt{2+a_{k+1}} > \sqrt{2+a_k} = a_{k+1}$

$\therefore$ " $a_{n+1} > a_n$ " True. i.e. $\{a_n\} \uparrow$

(c). Bounded + Monotone $\Rightarrow$ Convergent

Assume $\lim_{n\to\infty} a_n = L$, Then

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{\lim_{n\to\infty} 2+a_n}$

$\therefore$ $L = \sqrt{2+L}$

$L^2 - L - 2 = 0$

$(L+1)(L-2) = 0$

$\therefore$ $L = 2$ or $L = -1$

Since $a_n \geq a_0 = 0$ for all n

$\therefore$ $L = 2$ i.e. $\lim_{n\to\infty} a_n = 2$