Section 8.1 Problems

1. We have a theorem that states, “bounded, monotonic sequences must converge.” Determine if the following statements are true or false, and if false, give a counterexample:
   (a) If \( \{a_n\} \) is bounded, then it converges.
   (b) If \( \{a_n\} \) is not bounded, then it diverges.
   (c) If \( \{a_n\} \) diverges, then it is not bounded.

2. Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.
   
   (a) \( a_n = 10 + \left( \frac{-1}{9} \right)^n \)
   (b) \( b_n = \arctan \left( \frac{n^2 + 1}{n} \right) \)
   (c) \( c_n = \ln \left| \frac{2n + 1}{3n + 4} \right| \)
   (d) \( d_n = \frac{(-1)^n}{\sqrt{n}} \)
   (e) \( e_n = \frac{3 - 4^n}{2 + 7 \cdot 3^n} \)
   (f) \( f_n = \frac{8^{2n}}{n!} \)
   (g) \( g_n = \frac{\ln(n)^2}{n} \)
   (h) \( h_n = \left( \frac{5 - \frac{3}{n}}{2} \right)^{2n} \)
   (i) \( i_n = n\pi \cos(n\pi) \)
   (j) \( j_n = \frac{3^n}{2^n + 4^n} \)

3. Give an example of a divergent sequence \( a_n \) where \( \left\{ \frac{1}{a_n} \right\} \) converges.

4. Give an example of divergent sequences \( a_n \) and \( b_n \) where \( a_n + b_n \) converges.

5. Show that \( a_n = \frac{1}{2n + 1} \) is decreasing using induction or derivatives.

6. Show that \( b_n = \frac{3n^2}{n^2 + 1} \) is increasing using derivatives.

7. Is the sequence \( a_n = \frac{1 - 4^n}{2^n} \) bounded above? bounded below? monotonic?

8. Is the sequence defined recursively as \( a_1 = 1 \) and \( a_n = 2a_{n-1} - 3 \) bounded above? bounded below? monotonic?

9. Let \( \{a_n\} \) be defined recursively by \( a_0 = 0 \) and \( a_{n+1} = \sqrt{2 + a_n} \).
   (a) Show that \( a_n < 2 \) using induction.
   (b) Show that \( a_n \) is increasing by induction.
   (c) Show that \( a_n \) converges to 2 by assuming \( \lim_{n \to \infty} a_n = L \).